The surface of a contact discontinuity is defined from the equality $[P] = 0$. The remaining conditions (1.6) are satisfied by virtue of the equality $u_n = 0$. Hence, any two solutions of (5.1) with intersecting flow domains are coupled along a surface of the same pressure through a contact discontinuity.

REFERENCES

1. IBRAGIMOV N.KH., Classification of invariant solutions of the equations for the two-dimensional non-stationary motion of a gas, Prikl. Matem. i Tekh. Fiz., 4, 1966.
2. SEDOV L.I., Similarity and Dimensionality Methods in Mechanics, Nauka, Moscow, 1965.
3. OVSYANNIKOV L.V., Lectures on the Foundations of Gas Dynamics, Nauka, Moscow, 1981.
4. OVSYANNIKOV L.V., Group Analysis of Differential Equations, Nauka, Moscow, 1978.
5. LAPKO B.V., Construction of optimal systems of subgroups of the Lie transformation group which are permitted by the equations of gas dynamics, in: Dynamics of a Continuous Medium, Inst. Gidrodinamiki Sib. Otd. Akad. Nauk SSSR, Novosibirsk, 14, 1973.
6. BRYUNO A.D., The Local Method for the Non-Linear Analysis of Differential Equations, Nauka, Moscow, 1979.

Translated by E.L.S.

PMM U.S.S.R.,Vol.52,No.6,pp.761-766,1988
Printed in Great Britain

0021-8928/88 $10.00+0.00

EFFECT OF SPHERICALLY SYMMETRIC MASS FLOW FROM THE SURFACE OF A PARTICLE ON THE FORCE OF INTERACTION WITH A PLANE SURFACE*

A.M. GOLOVIN and A.F. ROGOVOI

A stationary velocity field of the flow of a gaseous medium generated by uniform radial injection from the surface of a spherical particle near a wall is considered in the Stokes' approximation. Bispherical coordinates are used to write the expression for the stream function. A formula is obtained for the force acting on the spherical particle when there is an arbitrary mass flow from its surface, generalizing earlier results /1, 2/. An expression for the force acting on the particle is obtained for the case of spherically symmetric injection from the surface of the particle, and asymptotic formulas at short and long distances from the wall are studied.

An analogous problem concerning the forces of interaction between two spherical particles of the same radius, when uniform injection of equal intensity takes place from their surfaces, is discussed. This is equivalent to the problem of the interaction of a spherical particle with a free surface. A general expression for the force of interaction, and its asymptotic forms for short and long distances, are obtained.

1. Formulation of the problem. Evaporation from a spherical particle near a solid or free surface, caused by various processes taking place in the gaseous medium, at the surface and inside of the particle, can be regarded in certain cases as being close to spherically symmetric.

Let us consider, for example, a particle with internal heat emission, situated near a wall at a uniform temperature T_w, equal to the temperature of the gaseous medium far from the particle. We shall assume that the heat flux from the surface of the particle is governed

**Prikl.Matem.Mekhan.*,52,6,976-981,1988

by molecular thermal conductivity and radiation, and we shall assume that the path length of the radiation considerably the distance from the centre of the particle to the wall. Let the thermal conductivity of the gas surrounding the particle be vanishingly small compared with the thermal conductivity of the particle. This means that the temperature gradients within the particle will be small compared with those occurring the gaseous medium.

In the quasilinear approximation the velocity of the mass flux from the surface of the particle will be given by the balance of the energy fluxes from its surface:

$$\rho L w = \varkappa \partial T/\partial n + \rho L w_0, \quad \rho L w_0 = {}^1/_3 \rho' k a + \varepsilon\sigma (T_w^4 - T_a^4)$$

Here ρ, ρ' is the density of the gaseous medium and the particle, respectively, L is the latent heat of evaporation, $\varkappa$ is the molecular thermal conductivity, $\mathbf{n}$ is the unit vector of outer normal to the element of the particle surface, k is the intensity of internal evolution of heat, a is the particle radius, T_a is its surface temperature, σ is the Stefan-Boltzmann constant and ε is the effective degree of blackness of the particle surface.

From this we see that in order to calculate the mass flux from the surface of the particle, we must solve the corresponding thermal problem, except in the cases when the internal heat emission is sufficiently intense or the radiation flux is considerable, in which case we can assume that $w \approx w_0$. The latter relation holds when

$$\rho L w_0 \gg (\varkappa/d) \mid T_w - T_a \mid$$

Here d is the characteristic scale of temperature variation near the particle, $d = a$ if $h \gg a$ and $d = h - a$ if $h - a \ll a$.

Analogous estimates can be obtained for a particle with internal heat emission near the free surface.

In a cylindrical system of coordinates, where the z axis passes through the centre $(z = h)$ of the spherical particle in a direction perpendicular to the wall $(z = 0)$, the stationary axisymmetric flow in the case of spatially homogeneous injection from the surface of the particle, is described in the Stokes approximation by the following equation and the corresponding boundary conditions:

$$D^4\psi = 0 \quad \left(D^2 = r\frac{\partial}{\partial r}\frac{1}{r}\frac{\partial}{\partial r} + \frac{\partial^2}{\partial z^2}\right) \tag{1.1}$$

$$r^2 + (z - h)^2 = a^2, \quad \mathbf{v} = w_0\mathbf{n}; \quad z = 0, \quad \mathbf{v} = 0$$

$$r^2 + z^2 \to \infty, \quad \mathbf{v} = 0$$

$$\left(v_r = -\frac{a^2 w_0}{r}\frac{\partial\psi}{\partial z}, \quad v_z = \frac{a^2 w_0}{r}\frac{\partial\psi}{\partial r}\right)$$

Let us introduce the bispherical coordinates ξ, η, φ connected with the cylindrical coordinates r, z, φ by the relations

$$\frac{z}{a} = \frac{c \operatorname{sh}\xi}{\operatorname{ch}\xi - \mu}, \quad \frac{r}{a} = \frac{c \sin\eta}{\operatorname{ch}\xi - \mu}, \quad \mu = \cos\eta$$

Here φ is the azimuthal angle unique for the cylindrical and bispherical systems of coordinates, and c is the parameter of the coordinate system found from the ratio of the distance h between the particle centre and the wall, and the radius a of the particle

$$c = \operatorname{sh}\xi_0, \quad h/a = \operatorname{ch}\xi_0$$

where $\xi = \xi_0\ (\xi_0 > 0)$ is the equation of the surface of the particle in the bispherical system of coordinates.

The boundary value problem (1.1) for determining the stream function becomes, in bispherical coordinates,

$$D^4\psi = 0 \tag{1.2}$$

$$\left(D^2 = \frac{\operatorname{ch}\xi - \mu}{c^2}\left[\frac{\partial}{\partial\xi}(\operatorname{ch}\xi - \mu)\frac{\partial}{\partial\xi} + (1 - \mu^2)\frac{\partial}{\partial\mu}(\operatorname{ch}\xi - \mu)\frac{\partial}{\partial\mu}\right]\right)$$

$$\psi(\xi_0, \mu) = -\frac{(\operatorname{ch}\xi_0 - 1)(1 + \mu)}{\operatorname{ch}\xi_0 - \mu}, \quad \frac{\partial\psi}{\partial\xi}(\xi_0, \mu) = 0$$

$$\psi(0, \mu) = \frac{\partial\psi}{\partial\xi}(0, \mu) = 0; \quad \xi \to 0, \quad \mu \to 1, \quad \psi = 0$$

The general solution of Eq.(1.2) is

$$\psi(\xi, \mu) = \frac{1}{(\operatorname{ch}\xi - \mu)^{3/2}}\sum_{n=0}^{\infty} U_n(\xi) V_n(\mu) \tag{1.3}$$

$$U_n(\xi) = A_n \operatorname{ch}\left(n - \frac{1}{2}\right)\xi + B_n \operatorname{sh}\left(n - \frac{1}{2}\right)\xi +$$
$$C_n \operatorname{ch}\left(n + \frac{3}{2}\right)\xi + D_n \operatorname{sh}\left(n + \frac{3}{2}\right)\xi$$
$$V_n(\mu) = P_{n-1}(\mu) - P_{n+1}(\mu) - 2\delta_{n0}, \quad \delta_{n0} = \begin{cases} 1, & n = 0 \\ 0, & n \neq 0 \end{cases}$$

Here A_n, B_n, C_n, D_n are arbitrary constants determined from the boundary conditions and $P_n(\mu)$ is the Legendre polynomial.

The function (1.3) differs from the known solution of the Stokes equations in bispherical coordinates /1, 2/ in the zeroth term, which is determined in such a manner that the stream function satisfies the condition $\psi = 0$ when $\eta = \pi$. This makes it possible to simplify the problem by choosing $\psi = 0$ when $\xi = 0$. It can be confirmed by direct substitution that when $V_0(\mu)$ is defined in this manner, the function $\psi(\xi, \mu)$ remains a solution of Eq.(1.2) when the values of the constants A_0, B_0, C_0, D_0 are arbitrary.

In the Stokes approximation the expression for the unique non-zero component of the force acting on the particle has the form /1, 2/

$$F_z = F = -\pi\rho v a w_0 \int r^3 \frac{\partial}{\partial n}\left(\frac{D^2\psi}{r^2}\right) ds \tag{1.4}$$

Here ds is the element of length of the contour of the meridian cross-section of the spherical particle. The integration over ds is carried out in the direction $[\mathbf{e}_\varphi \times \mathbf{n}]$, where $\mathbf{e}_\varphi$ is the unit vector chosen in the same manner as in the case of a cylindrical system of coordinates. The difference in sign in (1.4) as compared with /1, 2/ is the result of choosing a function ψ of different sign in the formulas which give the velocity components in terms of the stream function.

Changing in expression (1.4) to bispherical coordinates, we obtain

$$K = \frac{F}{\pi\rho v a w_0} = \int_{-1}^{1} \frac{d\mu}{(\operatorname{ch}\xi_0 - \mu)^3} \left\{\frac{\partial}{\partial\xi}\left[(\operatorname{ch}\xi - \mu)^2 D^2\psi\right]\right\}_{\xi_0} \tag{1.5}$$

Substituting the general expression for the stream function (1.3) into (1.5) we obtain the following result:

$$K = -\frac{2\sqrt{2}}{\operatorname{sh}\xi_0}\left[A_0\left(\frac{3}{\operatorname{ch}\xi_0 - 1} - 1\right) + (3D_0 - B_0)\operatorname{cth}\frac{\xi_0}{2} + \right. \tag{1.6}$$
$$\left. 3C_0\frac{\operatorname{ch}\xi_0}{\operatorname{ch}\xi_0 - 1} + \sum_{n=0}^{\infty}(2n+1)(A_n + B_n + C_n + D_n)\right]$$

Formula (1.6) represents a generalization of the well-known expression for the force acting on a particle in a viscous fluid /1, 2/ to the case of an arbitrary mass flux from its surface, and this results in the appearance of additional terms containing A_0, B_0, C_0 and D_0.

2. Determining the coefficients in the problem of the interaction between a particle and a plane surface. The stream function (1.3) must satisfy the boundary conditions of adhesion (1.2) at the plane wall. From this it follows that

$$A_n = -C_n, \quad B_n = -\frac{2n+3}{2n-1}D_n, \quad n \geqslant 0 \tag{2.1}$$

We obtain the remaining coefficients using the assumption (1.2) that we have a uniform radial injection from the surface of the spherical particle. In this case we have

$$\sum_{n=0}^{\infty} U_n(\xi_0)V_n(\mu) = -(\operatorname{ch}\xi_0 - 1)(1+\mu)\sqrt{\operatorname{ch}\xi_0 - \mu}$$
$$\sum_{n=0}^{\infty}\frac{dU_n}{d\xi}(\xi_0)V_n(\mu) = -\frac{3\operatorname{sh}\xi_0(\operatorname{ch}\xi_0 - 1)(1+\mu)}{2\sqrt{\operatorname{ch}\xi_0 - \mu}}$$

Differentiating these relations with respect to the variable μ, taking the relation
$$dV_n/d\mu = -(2n+1)P_n$$
into account, followed by an expansion of the right-hand sides of the resulting equations in

series in orthogonal Legendre polynomials, we obtain for each $n \geqslant 0$, a linear algebraic system of two equations for determining the coefficients A_n, B_n, and their solution has the form

$$\begin{aligned}
&A_n = G_n/\Delta_n, \quad B_n = H_n/\Delta_n, \quad E = \exp(-\xi_0) \qquad (2.2)\\
&G_n = -Q\,(p_n + q_n E^{2n}), \quad Q = {}^1/_4\sqrt{2}\,(2n+1)(\operatorname{ch}\xi_0 - 1)\\
&H_n = Q\,(f_n + g_n E^{2n}), \quad \Delta_n = \frac{(2n+1)^2}{2n+3}[4\operatorname{ch}(2n+1)\xi_0 -\\
&\quad (2n+1)^2\operatorname{ch}2\xi_0 + (2n-1)(2n+3)]\\
&p_n = \frac{6(2n+1)}{2n+3} + (4n+1)E - \frac{8n^2+2n-3}{2n+3}\frac{1}{E} +\\
&\quad \frac{4n(n+1)}{2n+3}\left(E^2 - \frac{1}{E^2}\right)\\
&q_n = (n+1)\left(2+\frac{1}{E}\right) + \frac{3(2n+1)}{2n+3}E - \frac{n(2n-1)}{2n+3}(2+E)E^2\\
&f_n = 2 + (4n+1)E - (4n+3)\frac{1}{E} +\\
&\quad 4n(n+1)\left(\frac{1}{2n+3}E^2 - \frac{1}{2n-1}\frac{1}{E^2}\right)\\
&g_n = (n+1)\left(2+\frac{1}{E}\right) + \frac{4n^2+4n+9}{(2n-1)(2n+3)}E - n(2+E)E^2
\end{aligned}$$

From this we obtain the following expressions for the zeroth coefficients:

$$A_0 = C_0 = 0, \quad B_0 = 3D_0 = 3/\sqrt{2}$$

When $n \geqslant 1$, we obtain, using (2.1) and (2.2), the following expression for the sum of the coefficients appearing in formula (1.6):

$$A_n + B_n + C_n + D_n = 4B_n/(2n+3)$$

The above sum enables us to calculate the dimensionless force of interaction between the particle and the wall

$$K = -\frac{8}{\operatorname{sh}\xi_0}\left(1 + \sqrt{2}\sum_{n=1}^{\infty}\frac{2n+1}{2n+3}B_n\right) \qquad (2.3)$$

A study of the behaviour of this expression at short and long distances from the wall compared with the radius of the particle, yields the following asymptotic expressions:

$$\begin{aligned}
&\xi_0 \ll 1, \quad K = K_0 = \frac{12}{\xi_0^2} = \frac{6}{h/a - 1} \qquad (2.4)\\
&\xi_0 \gg 1, \quad K = K_\infty = 18E^2\left(1 + \frac{9}{4}E\right) = \frac{9a^2}{2h^2}\left(1 + \frac{9a}{8h}\right)
\end{aligned}$$

The results of numerical calculations of the magnitude of the dimensionless force K acting on the particle and determined from formula (2.3), and the corresponding asymptotic formulas (2.4), as a function of ξ_0 or h/a, are given below:

$\xi_0 = 0.1$	0.5	1	1.5	2	2.5
$h/a = 1.005$	1.128	1.543	2.352	3.762	6.132
$K = 1170$	36.6	5.76	1.47	0.444	0.145
$K_0 = 1200$	48	$K_\infty = 4.45$	1.35	0.430	0.144

From this we see that the asymptotic formulas (2.4) can be used even in the region of moderate values of ξ_0. When $h/a - 1 \leqslant 10^{-3}$ or $h/a > 10$, the error of the asymptotic formulas does not exceed 1%.

3. Interaction of two spherical particles. In the case of two spherical particles of radius a whose centres are separated from each other by distance of $2h$, the mass fluxes from their surfaces will lead to interaction between the particles. If the injection from their surfaces is of equal intensity, then their investigation reduces to computing the velocity field of the particles near the free surface ($z = 0$). The corresponding boundary value problem will differ from (1.1) only in a single boundary condition

$$z = 0, \quad v_z = 0, \quad \partial v_r/\partial z = 0 \qquad (3.1)$$

We shall introduce the stream function ψ just as in the previous case. The stream function will be defined by the solution of problem (1.2), if we replace the condition on the wall at $\xi = 0$ by the condition on the free surface, which follows from (3.1),

$$\psi(0,\mu) = \frac{\partial^2\psi}{\partial\xi^2}(0,\mu) = 0$$

The change in the boundary conditions will lead to the fact that the relations

$$A_n = C_n = 0 \tag{3.2}$$

in the general solution (1.3) will hold for any $n \geqslant 0$. The remaining coefficients B_n and D_n will be determined, as before, from the boundary conditions at the surface of the spherical particle, as a result of solving the system of two algebraic equations for any $n \geqslant 0$. Hence we can obtain

$$\begin{aligned}
&B_n = M_n/d_n, \quad D_n = N_n/d_n \\
&M_n = -Q(\alpha_n - \beta_n E^{2n}), \quad N_n = {}^1/_2\, Q(\gamma_n - \delta_n E^{2n}) \\
&d_n = (2n+1)^2[\operatorname{sh}(2n+1)\xi_0 - (2n+1)\operatorname{sh}\xi_0 \operatorname{ch}\xi_0] \\
&\alpha_n = 2n + (4n+3)\frac{1}{E} + \frac{4n(n+1)}{2n-1}\frac{1}{E^2} \\
&\beta_n = \frac{2n^2-n-3}{2n-1}E + n(2+E)E^2 \\
&\gamma_n = 4(n+1) + 2(4n+1)E + \frac{8n(n+1)}{2n+3}E^2 \\
&\delta_n = 4(n+1) + 2(n+1)\frac{1}{E} + \frac{2n(2n+5)}{2n+3}E
\end{aligned} \tag{3.3}$$

In the special case when $n = 0$ we have $B_0 = 3/\sqrt{2},\ D_0 = 1\sqrt{2}$.
Substituting (3.2) and (3.3) into formula (1.6) we obtain the following expression:

$$K = -\frac{2\sqrt{2}}{\operatorname{sh}\xi_0}\Big[2\sqrt{2} + \sum_{n=1}^{\infty}(2n+1)(B_n + D_n)\Big] \tag{3.4}$$

Evaluating the terms in this sum, we obtain

$$B_n + D_n = Q(f_n - g_n E^{2n})/d_n$$

where f_n and g_n are determined from formulas (3.2) of the previous problem.
The asymptotic formulas for the force of interaction between two particles have the form

$$\begin{aligned}
&\xi_0 \ll 1, \quad K = K_0 = \frac{3}{\xi_0^2} = \frac{3}{2(h/a-1)} \\
&\xi_0 \gg 1, \quad K = K_\infty = 6E^2\Big(1 + \frac{3}{2}E\Big) = \frac{3a^2}{2h^2}\Big(1 + \frac{3a}{4h}\Big)
\end{aligned} \tag{3.5}$$

The results of computing the dimensionless force of interaction between the particles using (3.4) and the asymptotic formulas (3.5) are given below:

$\xi_0 = 0.1$	0.5	1	1.5	2
$K = 295$	9.1	1.52	0.417	0.134
$K_0 = 300$	12	$K_\infty = 1.26$	0.399	0.132

4. Discussion of the results. From formulas (2.3) and (3.4) it follows that the interaction of the particle with the wall, or of two particles with each other, is characterized, when there is a spherically symmetric flow from the surfaces of the particles, by repulsion forces. If the mass flow is directed towards the surfaces of the particles (condensation), then clearly we will have the forces of attraction. An approximation value of the first term of the asymptotic expansion (2.4) of the force acting on the particle when long distances separate the particle from the wall, can be found using the formulation of the general solution of the Stokes equation in spherical coordinates, with the origin at the centre of the particle, and in its mirror image relative to the wall, by restricting ourselves to terms containing the first three Gegenbauer polynomials. The loading term of the asymptotic expression (3.5) for the force of interaction between two particles at long distances is obvius, since it represents the Stokes formula in which the velocity of the flow impinging on the particle is the velocity generated by the second particle at a distance $2h$.

In the case of the asymptotic formula at short distances $(h/a - 1 \ll 1)$, the leading terms in formulas (2.4) and (3.5) can be obtained in the approximation of the hydrodynamic theory of lubrication. The first terms of the corresponding two- and three-term asymptotic expressions for the force acting on the spherical particle moving in a direction perpendicular to the plane wall or to the free surface /3, 4/, agree with formulas (2.4) and (3.5) by virtue of the **equivalence** of these problems within the approximation of the hydrodynamic theory of lubrication.

REFERENCES

1. STIMSON M. and JEFFERY G.B., The motion of two spheres in a viscous fluid, Proc. Roy. Soc. A. 111, 757, 1926.

2. HAPPEL G. and BRENNER G., Hydrodynamics at Small Reynolds Numbers. Mir, Moscow, 1976.
3. COOLEY M.D.A. and O'NEILL M.E., On the slow motion generated in a viscous fluid by the approach of a sphere to a plane wall or stationary sphere, Mathematika, 16, 1, 1969.
4. ZINCHENKO A.Z., Calculation of hydrodynamic interaction between drops at low Reynolds numbers. PMM 42, 5, 1978.

Translated by L.K.

PMM U.S.S.R.,Vol.52,No.6,pp.766-773,1988
Printed in Great Britain

0021-8928/88 $10.00+0.00

MASS TRANSFER IN A PULSATING BUBBLE*

V.S. BERMAN and A.D. POLYANIN

Mass transfer between a pulsating bubble and a surrounding medium at large and small Peclet numbers is considered. The dependence of the Sherwood number on time is found for an arbitrary periodic law of variation of the bubble radius. The case of sinusoidal oscillations is studied in detail.

1. Dynamics of a pulsating bubble. The spherically symmetric oscillations of a bubble under various conditions have been studied in many publications (e.g. /1-7/). Let us list here the fundamental properties of such motions, which will be of use later when analysing mass transfer in a pulsating bubble.

The radial component of the velocity of the fluid outside the bubble is described by the expression

$$v_r = R^2R'/r^2, \quad R' = dR/dt_* \tag{1.1}$$

Here r is the radial coordinate, t_* is the time and $R = R(t_*)$ is the law of motion of the bubble boundary, which can be found, under fairly general assumptions, by solving the differential equation /1-5/

$$\rho(RR'' + {}^3/_2R'^2) + 4\mu R'/R = g_*(R) + \varphi_*(t_*) \tag{1.2}$$

where μ and ρ is the dynamic viscosity and the density of the surrounding medium.

In order to complete the formulation of the problem we must supplement Eq.(1.2) by the initial conditions $R(0) = R_0, R'(0) = 0$ where R_0 is the initial radius of the bubble. (Sometimes a periodic solution of Eq.(1.2) has to be found).

The function g_* in (1.2) is usually chosen in the form /1-5/

$$g_*(R) = p_{g0}(R_0/R)^{3\gamma} - p_\infty - 2\sigma/R \tag{1.3}$$

where p_∞ is the static pressure at infinity, σ is the surface tension, γ is the ratio of the specific heats and p_{g0} is a constant whose dimensions are that of pressure.

In the case of thin elastic spherical shells (e.g. a rubber ball) oscillating in a liquid or gas, a linear function R /8/ must be subtracted from the right-hand side of the expression (1.3) when $\sigma = 0$.

In the case of forced oscillation of the bubble, φ_* in (1.2) is a T_*-periodic function and is responsible for the variation in the pressure field. In this case we can assumed without loss of generality that the following condition holds:

$$\langle\varphi_*\rangle \equiv \frac{1}{T_*}\int_0^{T_*}\varphi_*(t_*)\,dt_* = 0$$

---

*Prikl.Matem.Mekhan.,52,6,982-990,1988